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MARTINGALE INEQUALITIES IN REARRANGEMENT INVARIANT FUNCTION SPACES

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ABSTRACT

The Burkholder–Davis–Gundy equivalence of the square function and maximal function of a martingale is extended to the setting of rearrangement invariant function spaces.

1. Introduction

In the language of Banach space theory, an equivalence theorem of Burkholder, Davis and Gundy [BDG] states that if $L_\Phi = L_\Phi(0,1)$ is the Orlicz function space defined by a convex symmetric Orlicz function Φ which satisfies the Δ_2 condition at ∞ (i.e., for some constant C, $\Phi(2t) \leqq C\Phi(t)$ for all $t > 1$), then for any martingale $f = \{f_n\}_{n=1}^{\infty}$ defined on $(0,1)$, the L_Φ-norm of the square function of f is equivalent to the L_Φ-norm of the maximal function of f; that is, for some constant $K = K(C)$ and all martingales f on $(0,1)$,

$$K^{-1} \| \sup_n |f_n| \|_{L_\Phi} \leqq \left\| \left(\sum_{n=1}^{\infty} |f_n - f_{n-1}|^2 \right)^{1/2} \right\|_{L_\Phi} \leqq K \| \sup_n |f_n| \|_{L_\Phi}.$$

Here the norm is defined for x a measurable function on $(0,1)$ by

$$\| x \|_{L_\Phi} = \inf\{t > 0 : \mathbf{E}\Phi(x/t) \leqq 1\},$$

where $\mathbf{E}$ denotes expectation. From this point of view, it is natural to ask

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whether a similar norm equivalence is true for other rearrangement invariant normed function spaces. In Theorem 3 we answer this question by showing that if X is a rearrangement invariant normed function space on (0, 1), then the X-norm of the square function of every martingale on (0, 1) is equivalent to the X-norm of the maximal function of the martingale if and only if the upper Boyd index of X is finite. The upper Boyd index (see [LT]) of a rearrangement invariant function space X on (0, 1) or on $(0, \infty)$ is the smallest p such that for all $0 < c < 1$

$$\| D_c \|_X \leqq c^{1/p}.$$

Here D_c is the dilation operator defined for $x \in X$ by

$$D_c x(t) = x(t/c)$$

where $x(s) \equiv 0$ if $s > 1$ and X is a rearrangement invariant function space on (0, 1). There is a corresponding lower Boyd index. These indices play an important role in interpolation theory. In fact, Boyd's original interpolation theorem (see [LT, p. 145]) can be combined with known arguments (see [LT, pp. 50, 51, 175]) to give an easy deduction of Theorem 3 from the special case of $L_p(0, 1)$, $1 < p < \infty$, when the lower Boyd index of X is strictly larger than one and the upper Boyd index of X is finite. In this case one even gets that the X-norm of the martingale is equivalent to the X-norm of the square function of the martingale.

From the point of view of Banach space theory, finiteness of the upper Boyd index of X is a condition which insures that X is "far" from L_∞. In particular, it is easy to check that such an X contains $L_p(0, 1)$ for some $p < \infty$, which is a weaker "far from L_∞" condition used in [JS]. In the proof of one of the implications of Theorem 3, we use a geometric "far from L_∞" condition equivalent to the finiteness of the upper Boyd index proved by Lindenstrauss and Tzafriri [LT, p. 141]. The more important implication of Theorem 3, that the X-norm of the square function of every martingale on (0, 1) is equivalent to the X-norm of the maximal function of the martingale if the upper Boyd index of X is finite, is proved by following the usual proof of the Burkholder–Davis–Gundy theorem. The main new ingredient, Proposition 1, generalizes Neveu's proof [N] of the Burkholder–Davis–Gundy convex function inequality to the appropriate rearrangement invariant function space setting.

For background on concepts from Banach space theory we refer to the book [LT]; in particular, we follow the definition of rearrangement invariant function space used there. The expository paper [B] is a good reference for background on the Burkholder–Davis–Gundy theorem.

2. The main result

We begin with the generalization of Neveu's result [N] mentioned in the introduction.

PROPOSITION 1. *Let X be a rearrangement invariant function space on* $(0, 1)$ *with upper Boyd index $p < \infty$. Let W and Z be nonnegative measurable functions on* $(0, 1)$ *such that for some $C > 1$*

$$\int_{[W>C\lambda]} W \leqq \int_{[W>\lambda]} Z \qquad \text{for all } \lambda > 0. \tag{1}$$

Then

$$\| W \|_X \leqq K \| Z \|_X, \tag{2}$$

where $K = K(p, C)$ depends only on p and C.

PROOF. First we replace W with a function which takes on only countably many values. Set

$$U = \sum_{k=-\infty}^{\infty} C^{k+1} 1_{[C^k < W \leq C^{k+1}]}$$

and notice that

$$C^{-1} U \leqq W \leqq U$$

and that, for $\lambda = C^k$,

$$[U > \lambda] = [W > \lambda].$$

Consequently,

$$\| W \|_X \leqq \| U \|_X \leqq C \| W \|_X \tag{3}$$

and, for $\lambda = C^k$,

$$\int_{[U>C\lambda]} U \leqq C \int_{[W>C\lambda]} W \leqq C \int_{[W>\lambda]} Z = C \int_{[U>\lambda]} Z; \tag{4}$$

that is, (U, Z) satisfies an inequality analogous to (1) and it is enough to prove that $\| U \|_X \leqq K \| Z \|_X$.

Next we observe that without loss of generality we may assume that U and Z are nonincreasing. Indeed, since U takes on only countably many values, there is a measure preserving transformation f: $(0, 1) \to (0, 1)$ such that $U^*(t) \equiv U(f(t))$ is nonincreasing (we use the notation U^* instead of the more common

U^* for the decreasing rearrangement of U because we reserve "*" for the maximal function of a martingale). We replace U by $U^\#$ and Z by $Z \circ f$; this leaves (4) unchanged and of course $\| U \|_X = \| U^\# \|_X$. If we now replace $Z \circ f$ by its decreasing rearrangement $Z^\#$, this can only increase the right hand side of (4) and $\| Z \|_X = \| Z^\# \|_X$.

Let $1 > \alpha > 0$ be a number which will be specified later and let μ denote Lebesgue measure. We partition the set $\mathbf{Z}$ of integers in the following way: $\mathbf{Z}_1$ consists of those k for which the interval $[U = C^{k+1}]$ has larger measure than $\alpha\mu[U = C^k]$. For $n \geqq 1$, define by recursion

$$\mathbf{Z}_{n+1} = \{k \in \mathbf{Z} : k - 1 \in \mathbf{Z}_n \text{ and } \mu[U = C^{k+1}] \leqq \alpha\mu[U = C^k]\}.$$

Set

$$U_n = U\left(\sum_{k \in \mathbf{Z}_n} 1_{[U = C^{k+1}]}\right)$$

and note that $U_n^\#$, the decreasing rearrangement of U_n, satisfies for $n = 2, 3, \ldots$

$$U_n^\# \leqq C^{n-1} D_{\alpha^{n-1}}(U_1^\#).$$

Consequently,

$$\| U_n \|_X \leqq (C\alpha^{1/p})^{n-1} \| U_1 \|_X$$

and, if α is such that $\sum_{n=2}^\infty (C\alpha^{1/p})^{n-1} = 1$; i.e. $\alpha = (2C)^{-p}$, then, since $U = \sum_{n=1}^\infty U_n$, we get

$$\| U \|_X \leqq 2 \| U_1 \|_X. \tag{5}$$

We would like to replace U by U_1, but U_1 is not decreasing, so we let V be the smallest nonincreasing function which is pointwise $\geqq U_1$ and work with V instead. V and U_1 take on the same nonzero values, namely C^{k+1} for k in $\mathbf{Z}_1$, and $V \leqq U$. The advantage of V over U is that for $\lambda = C^k$ with $k \in \mathbf{Z}_1$, if $[V > C\lambda] \neq [V > \lambda]$ then necesssarily $k + 1 \in \mathbf{Z}_1$ and then

$$\mu[V > C\lambda] > \frac{\alpha}{1+\alpha}\mu[V > \lambda].$$

We thus get that for $\lambda = C^k$ with $k \in \mathbf{Z}_1$, if $[V > C\lambda] \neq [V > \lambda]$ then

$$\int_{[V>\lambda]} V \leqq \frac{1+\alpha}{\alpha}\int_{[V>C\lambda]} V \leqq \frac{1+\alpha}{\alpha}\int_{[U>C\lambda]} U. \tag{6}$$

These inequalities also hold if $[V > C\lambda] = [V > \lambda]$, so for $\lambda = C^k$ with $k \in \mathbf{Z}_1$, we have from (6), (4), and the fact that $[V > \lambda] = [U > \lambda]$ that

(7) $$\int_{[V>\lambda]} V \leqq C\frac{1+\alpha}{\alpha}\int_{[V>\lambda]} Z.$$

Since every set of the form $[V>\lambda]$ for $\lambda>0$ is also of the form $[V>C^k]$ for an appropriate $k\in\mathbf{Z}_1$, (7) holds for all $\lambda>0$. This implies (see [LT, p. 125]) that

$$\| V \|_X \leqq C\,\frac{1+\alpha}{\alpha}\,\| Z \|_X,$$

so by (3) and (5) we get

$$\| W \|_X \leqq 2C\frac{1+\alpha}{\alpha}\,\| Z \|_X = 2C([2C]^p+1)\,\| Z \|_X.$$

Proposition 1 and Neveu's proof [N] of the convex function inequality yield:

COROLLARY 2. *Let X be a rearrangement invariant function space on* (0, 1) *with upper Boyd index* $p<\infty$. *Let* $\mathscr{A}_0\subset\mathscr{A}_1\subset\cdots$ *be an increasing sequence of sub σ-fields of the Borel σ-field and let* $z_1, z_2, \ldots$ *be a sequence of nonnegative measurable functions. Then*

$$\left\| \sum_{n=1}^{\infty} \mathbf{E}(z_n \mid \mathscr{A}_{n-1}) \right\|_X \leqq K \left\| \sum_{n=1}^{\infty} z_n \right\|_X,$$

where $K=K(p)$ *depends only on* p.

PROOF. Neveu [N, p. 175] proves that $W\equiv\Sigma_{n=1}^{\infty}\mathbf{E}(z_n \mid \mathscr{A}_{n-1})$ and $Z\equiv 2\,\Sigma_{n=1}^{\infty} z_n$ satisfy

$$\int_{[W>\lambda]} (W-\lambda) \leqq \frac{1}{2}\int_{[W>\lambda]} Z \qquad \text{for all } \lambda>0.$$

But then for all $\lambda>0$.

$$\int_{[W>2\lambda]} W \leqq 2\int_{[W>2\lambda]} (W-\lambda) \leqq 2\int_{[W>\lambda]} (W-\lambda) \leqq \int_{[W>\lambda]} Z,$$

so (W, Z) satisfies condition (1) with $C=2$. Proposition 1 yields the desired conclusion.

We are now ready for the main theorem. Given a martingale $(f_1, f_2, \ldots)$, we denote the square function $(\Sigma_{n=1}^{\infty} |f_n - f_{n-1}|^2)^{1/2}$ of $(f_n)_{n=1}^{\infty}$ by $S(f)$ and the maximal function $\sup_n |f_n|$ of $(f_n)_{n=1}^{\infty}$ by f^*.

Theorem 3. *Let X be a rearrangement invariant function space on* $(0, 1)$ *with upper Boyd index $p < \infty$ and let* $(f_1, f_2, \dots)$ *be a martingale on* $(0, 1)$. *Then*

$$c \| S(f) \|_X \leqq \| f^* \|_X \leqq C \| S(f) \|_X \tag{8}$$

where $0 < c < C < \infty$ depend only on p. Conversely, if either the right side of (8) *or the left side of* (8) *holds for all martingales on* $(0, 1)$, *then the upper Boyd index of X is finite.*

Proof. The proof of the first part follows closely Burkholder's proof of Theorem 15.1 of [B]. We need an analogue of Lemma 7.1 of [B] for rearrangement invariant function spaces.

Lemma 4. *Let x and y be nonnegative random variables on* $(0, 1)$ *and let X be a rearrangement invariant function space on* $(0, 1)$ *with upper Boyd index $p < \infty$. Suppose that $\beta > 0$, $\delta > 0$, $\varepsilon > 0$ satisfy $\beta \varepsilon^{1/p} < 1$ and*

$$\mu[y > \beta\lambda \text{ and } x \leqq \delta\lambda] \leqq \varepsilon\mu[y > \lambda] \qquad \text{for all } \lambda > 0.$$

Then

$$\| y \|_X \leqq \frac{\beta}{\delta(1 - \beta\varepsilon^{1/p})} \| x \|_X.$$

Proof. It is easy to check (see the proof of Lemma 7.1 in [B]) that

$$\mu[y > \beta\lambda] \leqq \varepsilon\mu[y > \lambda] + \mu[x > \delta\lambda].$$

Since $\varepsilon\mu[y > \lambda] = \mu[D_\varepsilon y > \lambda]$, we get that

$$\| \beta^{-1} y \|_X \leqq \| D_\varepsilon y \|_X + \| \delta^{-1} x \|_X \leqq \varepsilon^{1/p} \| y \|_X + \delta^{-1} \| x \|_X$$

and thus

$$\| y \|_X \leqq \frac{\beta}{\delta(1 - \beta\varepsilon^{1/p})} \| x \|_X.$$

The rest of the proof of the first part of Theorem 3 is a straightforward adaption of the proof of Theorem 15.1 of [B]; we leave the details to the reader.

Assume now that the upper Boyd index of X is infinite. For the second part of Theorem 3 we use Proposition 2.b.7 of [LT, p. 141], which states that for each $n \in \mathbb{N}$ and $\varepsilon > 0$ there are disjoint random variables $x_1, x_2, \dots, x_{2^n}$ on $(0, 1)$, all having the same distribution, which satisfy for all scalars $\{a_i\}_{i=1}^n$

$$\max_{1 \leqq i \leqq 2^n} |a_i| \leqq \left\| \sum_{i=1}^{2^n} a_i x_i \right\|_X \leqq (1 + \varepsilon) \max_{1 \leqq i \leqq 2^n} |a_i|.$$

Let $d_0, d_1, \ldots, d_n$ be the "Rademacher functions" over the x_i's; i.e., for $0 \leqq k \leqq n$

$$d_k = \sum_{j=1}^{2^k} (-1)^{j-1} \sum_{i=(j-1)2^{n-k}+1}^{j2^{n-k}} x_i .$$

Then $\{d_k\}_{k=0}^{n}$ forms a martingale difference sequence,

$$\left(\sum_{k=0}^{n} d_k\right)^* \geqq (n+1)|x_1|$$

and

$$S\left(\sum_{k=0}^{n} d_k\right) = \sqrt{n+1} \sum_{i=1}^{2^n} |x_i| .$$

Consequently,

$$\left\| \left(\sum_{k=0}^{n} d_k\right)^* \right\|_X \geqq n+1$$

and

$$\left\| S\left(\sum_{k=0}^{n} d_k\right) \right\|_X \leqq (1+\varepsilon)\sqrt{n+1}.$$

This shows that the right side of (8) does not hold with a uniform constant C for all finite length martingales. Standard reasoning now yields a martingale for which the right side of (8) does not hold for any constant C.

To show that the left side of (8) does not hold, we use the "double or nothing" martingale over the x_i's; i.e., for $1 \leqq k \leqq n$,

$$e_k = \sum_{i=1}^{2^{n-k}} x_i - \sum_{i=2^{n-k}+1}^{2^{n-k+1}} x_i .$$

Then $\{e_i\}_{i=1}^{n}$ forms a martingale difference sequence,

$$\left| \sum_{i=1}^{k} (-1)^{i-1} e_i \right| \leqq 2 \sum_{i=1}^{2^n} |x_i| \qquad \text{for } k = 1, 2, \ldots, n,$$

and hence

$$\left\| \left(\sum_{i=1}^{n} (-1)^{i-1} e_i\right)^* \right\|_X \leqq 2(1+\varepsilon).$$

But

$$\left\| S\left(\sum_{i=1}^{n} (-1)^{i-1} e_i\right) \right\|_X \geqq \sqrt{n}\, \| x_1 \|_X \geqq \sqrt{n}.$$

This completes the proof of Theorem 3.

REMARK 5. When specialized to the case where $X = L_\Phi$, inequality (8) states:

$$\text{“}\mathbf{E}\Phi(f^*) = 1 \Rightarrow \mathbf{E}\Phi(cS(f)) \leqq 1 \quad \textit{and} \quad \mathbf{E}\Phi(S(f)) = 1 \Rightarrow \mathbf{E}\Phi(C^{-1}f^*) \leqq 1,$$

$$\textit{where } 0 < c < C < \infty \textit{ depend only on the } \Delta_2 \textit{ constant of } \Phi\text{”},$$

which is formally weaker than the Burkholder–Davis–Gundy theorem. However, the full theorem can be easily recaptured from Theorem 3 by using an observation in Section 5 of [JS].

REMARK 6. It is easy to see that the upper and lower Boyd indices of the Lorentz space $L_{p,\infty}$ are both equal to p. The space $L_{p,\infty}$ is a normed space when $1 < p < \infty$, so from the easy version of Theorem 3 alluded to in the introduction, we get that for each $1 < p < \infty$ and martingale f on (0, 1),

$$\sup_{t>0} t^p \mu[S(f) > t] \approx \sup_{t>0} t^p \mu[f^* > t] \approx \sup_{t>0} t^p \mu[|f| > t]$$

where the constant of equivalence depends only on p. Although we have not seen this family of inequalities written down previously, we assume that they must be known.

REMARK 7. If X is a rearrangement invariant normed function space on (0, 1) which contains $L_p(0, 1)$ for some $p < \infty$ and $f = (f_1, f_2, \dots, f_n)$ is a martingale on (0, 1) which has *independent* increments, then it follows from the main result of [JS] that

$$\| S(f) \|_X \approx \| f \|_X \approx \| f^* \|_X,$$

where the constant of equivalence in the first equivalence depends only on p and on the norm of the formal identity mapping from $L_p(0, 1)$ into X (the constant of equivalence in the second equivalence is absolute; this follows via interpolation from the special cases of $X = L_1$ and $X = L_\infty$).

REMARK 8. Part of the impetus for the investigation reported on here came from de la Peña's paper [dlP]. See Section 5 of [JS] for the application which motivated us.

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